AD 638147

GEOPHYSICAL RESEARCH PAPERS

No. 3

DIFFRACTION EFFECTS IN THE PROPAGATION OF COMPRESSIONAL WAVES IN THE ATMOSPHERE

BY NORMAN A. HASKELL

Distribution of this document is unlimited.

March 1950

Base Directorate for Geophysical Research
Air Force Cambridge Research Laboratories
Cambridge, Massachusetts

GEOPHYSICAL RESEARCH PAPERS
No. 3

DIFFRACTION EFFECTS IN THE PROPAGATION OF COMPRESSIONAL WAVES IN THE ATMOSPHERE

BY NORMAN A. HASKELL

March 1950

Base Directorate for Geophysical Research
Air Force Cambridge Research Laboratories
Cambridge, Massachusetts

ABSTRACT

Asymptotic methods are used to find approximate solutions of the acoustic wave equation in a medium where the velocity is a continuously variable function of one coordinate. It is shown that, when the velocity function has a minimum, undamped normal mode solutions exist, and that such solutions are closely analogous to the internally reflected waves in the case of a medium made up of discrete layers. By converting the sum of the high-order normal modes into an equivalent integral, it is shown that superposition of these modes leads to geometrical ray theory modified by diffraction in a manner that may be computed from the incomplete Fresnel and Airy integrals.

CONTENTS

DIFFRACTION EFFECTS IN THE PROPAGATION OF COMPRESSIONAL WAVES IN THE ATMOSPHERE*

INTRODUCTION

Following F. J. W. Whipple's explanation of the zones of silence and of abnormal audibility of sounds from large explosions as an effect produced by the temperature structure of the atmosphere, a number of estimates of the temperature versus altitude function at altitudes in the range from 25 to 60 km have been made[2] on the basis of travel time measurements. As in the analogous problem in seismology, the determination of the velocity-altitude function from the observed time-distance curve has been based on ray geometry. The ray concept is, of course, valid and useful if the frequencies involved are sufficiently high, but it does not, in itself, provide any criterion for determining how high the frequency must be. In this report we shall consider diffraction effects by applying a method suggested by Pekeris[3] to find an approximate solution of the wave equation for the case in which the velocity is a function of altitude only and has a single minimum. Although the discussion will be carried through with particular reference to propagation in the atmosphere, it will be apparent that the method is equally applicable to the long range sound transmission in the ocean which is associated with the existence of a velocity minimum ("Sofar" channel) and, with some modification, to the propagation of seismic waves in cases where a low velocity layer is present.

*Manuscript received 10 January 1950

[1] F. J. W. Whipple, *Nature*, vol III, p 187; 1923.

[2] B. Gutenberg, *Handbuch der Geophysik*, vol 9, p 89; 1932. A summary of the earlier work done on this subject is given here.
E. F. Cox, *J. Acoust. Soc. Am.*; vol 19, p 832, 1947; vol 21, p 6, 1949. Recent results from the Arco, Idaho and Helgoland blasts are discussed here.

[3] C. L. Pekeris, "Theory of Propagation of Sound in a Half Space of Variable Sound Velocity," *J. Acoust. Soc. Am.*, vol 18, p 295; 1946.

1. WAVE EQUATION

When the effect of gravity and the consequent variation of initial density are not ignored, the hydrodynamic equations of motion generally do not possess irrotational solutions that can be expressed in terms of a velocity potential.[4] If the relative change of the initial density within a wavelength is small, however, the motion may be approximated *locally* by a velocity potential ϕ satisfying the simple wave equation,

$$c^2(z)\nabla^2\phi - \partial^2\phi/\partial t^2 = 0 \quad , \tag{1·01}$$

with the particle velocity vector V and the pressure perturbation p given by

$$V = - \operatorname{grad} \phi \tag{1·02}$$

$$p = \rho(z) \frac{\partial\phi}{\partial t} \quad , \tag{1·03}$$

where $c(z)$ is the local velocity of sound and $\rho(z)$ is the local density, both functions of the altitude z. If these equations are applied to a case in which the range of altitudes involved is large, it is found that the conservation of energy is violated. The acoustic energy flux density is

$$J = pV = -\rho(z) \frac{\partial\phi}{\partial t} \operatorname{grad} \phi \quad . \tag{1·04}$$

Since ϕ is a solution of Eq. (1·01), the factor $-(\partial\phi/\partial t)\operatorname{grad}\phi$ in Eq. (1·04) expresses the effect of geometrical ray divergence or convergence; and the factor $\rho(z)$ then leads to a loss of energy by a wave travelling in the direction of decreasing density, and a gain of energy by a wave travelling in the direction of increasing density. Thus, even though the terms depending on gravity and initial density variation are small, their cumulative effect may be very large when the range of altitudes considered is great. We may restore the conservation of energy, however, while retaining the simplicity of Eq. (1·01), by multiplying the right-hand sides of Eqs. (1·02) and (1·03) by a dimensionless quantity $f(z)$ which is a slowly varying function of z in the sense that the relative change in $f(z)$ within a wavelength is small. The equations of motion and continuity then are satisfied locally to the same order of approximation as that involved in deriving Eqs. (1·01), (1·02), and (1·03). If we set $f(z) = [\rho(h)/\rho(z)]^{1/2}$ where $\rho(h)$ is the density at the altitude, h, of the source, Eqs. (1·02), (1·03), and (1·04) become

[4] H. Lamb, *Hydrodynamics* 6th ed.; Cambridge University Press: pp 541-556.

$$V = -\left[\frac{\rho(h)}{\rho(z)}\right]^{1/2} \text{grad } \phi \qquad (1\cdot05)$$

$$p = [\rho(h)\rho(z)]^{1/2} \frac{\partial \phi}{\partial t} \qquad (1\cdot06)$$

$$J = -\rho(h) \frac{\partial \phi}{\partial t} \text{ grad } \phi \quad . \qquad (1\cdot07)$$

Since J, as given by Eq. (1·07), no longer depends on the altitude except through the effect of convergence or divergence expressed by the function ϕ, the energy is conserved at all altitudes; also, it is evident from the definition that if $\rho(z)$ is a slowly varying function of altitude, $f(z)$ is also a slowly varying function. The condition for the validity of Eqs. (1·01), (1·05), and (1·06) then may be expressed generally by the inequality

$$\left|\frac{\lambda}{2\pi} \frac{d(\log \rho)}{dz}\right| \ll 1 \quad , \qquad (1\cdot08)$$

where λ is the wavelength. The average value of $d \log \rho/dz$ to an altitude of 50 km in the atmosphere is about -0.14 km^{-1} so that Eq. (1·08) is equivalent to $\lambda \ll 45$ km, which implies that the frequency must be $\nu \gg 0.0067$ cycle per second. The approximate methods that we shall apply to the solution of Eq. (1·01), however, will provide a somewhat more severe low-frequency limitation than this.

2. NORMAL MODE SOLUTIONS OF THE WAVE EQUATION

Periodic solutions of Eq. (1·01) in a cylindrical coordinate system with axis passing through the source may be written in the form

$$\phi(r,z,t,k) = e^{i\omega t} J_0(kr) F(z,k) \quad , \qquad (2\cdot01)$$

where $F(z,k)$ satisfies the equation

$$\frac{d^2F}{dz^2} + F\left[\frac{\omega^2}{c^2(z)} - k^2\right] = 0 \quad . \qquad (2\cdot02)$$

The boundary conditions are

(1) At the ground surface $(z = 0)$ the vertical component of the particle velocity vanishes.

(2) $V(z) \to 0$ as $z \to \infty$.

(3) In the neighborhood of the source $(r = 0, \; z = h)$ the integrated solution

$$\phi(r,z,t) = \int_0^\infty \phi(r,z,t,k)\, dk \tag{2·03}$$

must reduce to the value appropriate to a point source, namely,

$$\phi \to \frac{A \exp\{i\omega[t - (1/c)\sqrt{r^2 + (z-h)^2}]\}}{\sqrt{r^2 + (z-h)^2}} \quad . \tag{2·04}$$

Designating the regions below and above the source by subscripts 1 and 2, respectively, condition (1) is

$$\left.\frac{dF_1}{dz}\right|_0 = 0 \quad ; \tag{2·05}$$

condition (2) requires that F_2 be a particular solution of Eq. (2·02) such that, with the factor $e^{i\omega t}$, it represents an upward travelling wave at large values of z; and condition (3) is satisfied[3] if we impose the conditions at $z = h$

$$F_1(h,k) = F_2(h,k) \tag{2·06}$$

$$\left.\frac{dF_1}{dz}\right|_h - \left.\frac{dF_2}{dz}\right|_h = 2Ak \quad . \tag{2·07}$$

Let $M(z,k)$ and $N(z,k)$ be two linearly independent solutions of Eq. (2·02) that behave asymptotically for large values of z (with the factor $e^{i\omega t}$) like downward and upward travelling waves, respectively. The boundary conditions are then satisfied by

$$F_1(z,k) = \frac{2AkN(h,k)}{b}\left\{\frac{N(z,k)M'(0,k)}{N'(0,k)} - M(z,k)\right\} \tag{2·08}$$

and

$$F_2(z,k) = \frac{2AkN(z,k)}{b}\left\{\frac{N(h,k)M'(0,k)}{N'(0,k)} - M(h,k)\right\} \quad , \tag{2·09}$$

where the primes denote differentiation with respect to z, and $b = MN' - M'N$ is a constant by virtue of the Abelian identity which holds between any two linearly independent solutions of Eq. (2·02).

We now assume that the velocity versus altitude function has the general character shown in Fig. 1. This function and its first derivative are taken to be continuous. There is a variety of evidence for the existence in the atmosphere of a second temperature

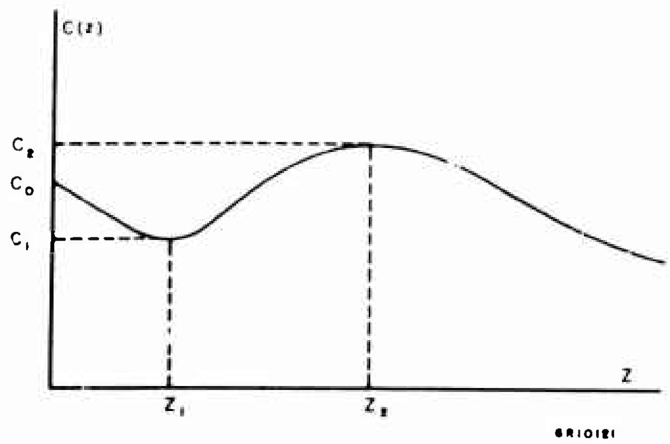

Fig. 1. General character of assumed velocity-altitude function.

minimum at an altitude of approximately 80 km, followed by a rise to indefinite heights. At very great altitudes, however, attenuation, which is not allowed for in Eq. (1·09), becomes increasingly important--and the more so the higher the frequency. Consequently, an appreciable return of wave energy to lower levels by refraction from altitudes above the first maximum is not to be expected except for very low frequencies. The assumption of a monotonically decreasing velocity above the altitude z_2, as indicated in Fig. 1, thus may be regarded merely as a device for ensuring that any wave energy that has travelled above the altitude z_2 will not be returned to lower levels. The method which will be elaborated below can, in principle, be applied to a velocity function with any number of minima and maxima, but for the present we shall confine our attention to functions with only one minimum and one maximum.

Pekeris[3,5] has shown that the integral (2·03) for ϕ can be reduced by replacing $J_0(kr)$ by $[H_0^{(1)}(kr) + H_0^{(2)}(kr)]/2$ followed by a transformation of the path of integration in the complex k-plane, to a sum of residues taken at the zeros of $N'(0,k)$ plus certain integrals around cuts in the k-plane. The contribution to ϕ of the residues, i.e., the "normal mode" solutions, is

$$\phi_{res} = -2\pi i A e^{i\omega t} \sum_n \frac{k_n H_0^{(2)}(k_n r) N(z,k_n) N(h,k_n)}{N(0,k_n)\,[\partial N'(0,k)/\partial k]_{k=k_n}}, \qquad (2\cdot 10)$$

where the k_n's are the roots of $N'(0,k_n) = 0$ and the constant b has been eliminated by use of the relation $b = -N(0,k_n)M'(0,k_n)$. Since Eq. (2·10) is unaltered by interchanging z and h, the expression is applicable both above and below the level of the source.

[5] C. L. Pekeris, "Theory of propagation of explosive sound in shallow water," *Geol. Soc. Am. Memoir*, vol 27; 1948 (*Propagation of Sound in the Ocean*).

At large distances from the source the radial factor in Eq. (2·10) behaves like

$$H_0{}^2(k_n r) \longrightarrow \left[\frac{2}{\pi k_n r}\right]^{1/2} \exp[-\,i(k_n r - \frac{\pi}{4})] \quad . \tag{2·11}$$

In conjunction with the factor $e^{i\omega t}$ this must represent a progressive wave travelling in the direction of increasing r, so that the real part of k_n must be positive and the imaginary part negative or zero. Those roots that have a finite imaginary part (if any exist) lead to an exponential decay factor, so that for large values of r the significant terms in Eq. (2·10) are those for which k_n is real.

3. APPROXIMATE EXPRESSIONS FOR THE FUNCTION N(z,k)

We introduce the parameter $\beta(k) = \omega/k$ and write

$$Q(z,k) = \begin{cases} +\,k\sqrt{\left[\dfrac{\beta(k)}{c(z)}\right]^2 - 1} & \beta(k) > c(z) \\[2ex] -\,ik\sqrt{1 - \left[\dfrac{\beta(k)}{c(z)}\right]^2} & \beta(k) < c(z) \end{cases} \tag{3·01}$$

$$u(z,k) = \int_a^z Q\,dz \quad , \tag{3·02}$$

where the lower limit of integration, a, remains to be defined. Approximate solutions of Eq. (2·02), asymptotically valid for sufficiently high frequencies, may be written in the form[6]

$$F = \left[\frac{u}{Q}\right]^{1/2} C(u) \quad , \tag{3·03}$$

where C is a linear combination of Bessel functions of order 1/3. The differential equation for which Eq. (3·03) is an exact solution is

$$F'' + (Q^2 - S''/S)\,F = 0 \quad , \tag{3·04}$$

[6] R. E. Langer, "On the Connection Formulae and the Solutions of the Wave Equation," *Phys. Rev.*; vol 51, p 669; 1937.

where

$$S = u^{1/6} Q^{-1/2} \tag{3·05}$$

and

$$S''/S = \tfrac{3}{4}(Q'/Q)^2 - \tfrac{1}{2}(Q''/Q) - \tfrac{5}{36}(Q/u)^2 \quad . \tag{3·06}$$

Thus Eq. (3·03) is an approximate solution of Eq. (2·02) for values of z such that S''/S is small compared to Q^2. For a given value of $\beta(k)$, Q and u are proportional to ω, so that S''/S is independent of ω. Thus it is always possible to find a value of ω great enough to make $Q^2 >> S''/S$ except at points where $Q^2 = 0$ or S''/S becomes infinite. However, if we take the integration constant, a, at a first-order zero of Q^2, it can be shown from Eq. (3·06) that S''/S remains finite at $z = a$, so that this zero of Q is not a singular point of Eq. (3·04). Thus an expression of the form (3·03), which is a valid approximation to a particular solution of Eq. (2·02) for $z > a$, represents the same particular solution for $z < a$. If Q has more than one zero, say, $z = a_1$ and $z = a_2$, the range of validity of the solution obtained by setting $a = a_1$ in Eq. (3·02) does not extend beyond $z = a_2$, since $u(a_2)$ is not zero and S''/S becomes infinite at a_2. However, another solution valid on both sides of a_2 is obtained by setting $a = a_2$ in Eq. (3·02). We then have two approximate solutions, both valid in the range $a_1 < z < a_2$: namely,

$$F_1 = \left(\frac{u_1}{Q}\right)^{1/2} C_1(u_1)$$

and

$$F_2 = \left(\frac{u_2}{Q}\right)^{1/2} C_2(u_2) \quad ,$$

where u_1 and u_2 are the values of u obtained by setting $a = a_1$ and a_2, respectively, in Eq. (3·02), and C_1 and C_2 may be different combinations of Bessel functions of order 1/3. So that both F_1 and F_2 may be asymptotic representations of the same solution F of Eq. (2·02), it is evidently necessary that there be a connection between the coefficients of the Bessel functions in C_1 and C_2. This connection is readily established by noting that the leading term in the asymptotic expansion of $C(u)$ has the form

$$C(u) \longrightarrow \left(\frac{2}{\pi u}\right)^{1/2} (Ae^{iu} + Be^{-iu}) \quad ,$$

where A and B are constants (in general complex). F_1 and F_2 then will be asymptotically equivalent in the range $a_1 < z < a_2$ if their respective coefficients are connected by the relations

$$A_2 = A_1 \exp i(u_1 - u_2)$$

and (3·07)

$$B_2 = B_1 \exp[-i(u_1 - u_2)]$$

where

$$u_1 - u_2 = \int_{a_1}^{a_2} Q\, dz$$

is a constant.

For certain values of $\beta(k)$ the zeros of Q^2 may be of higher order than the first; for example, when $\beta(k) = c_1$ or c_2 (Fig. 1), the zeros of Q^2 at $z = z_1$ or z_2 are of the second order. In this case the asymptotic solutions must be expressed in terms of Bessel functions of order 1/4 in place of those of order 1/3; however, these special cases need not be considered here.

The quantity S''/S also becomes infinite if either $c'(z)$ or $c''(z)$ becomes infinite, that is, if there is a discontinuity in either the velocity or the velocity gradient. In this case Eq. (3·03) will not represent the same solution of Eq. (2·02) on both sides of the discontinuity; that is, we must allow the coefficients of the Bessel functions to change discontinuously at a discontinuity in c or c', which corresponds physically to the existence of a reflected wave. We do not wish to complicate the present discussion by a consideration of such effects, and it is for this reason that the continuity of c and c' was assumed above.

In the particular case under consideration, we shall use subscripts 1, 2, and 3 to distinguish the parts of the velocity-altitude function in the ranges 0 to z_1, z_1 to z_2, and z_2 to infinity, respectively. When $\beta(k) < c_1$, Q has one zero, $z = a_3$; when $c_1 < \beta(k) < c_0$, it has three zeros, a_1, a_2, and a_3; when $c_0 < \beta(k) < c_2$, it has two zeros, a_2 and a_3; and when $\beta(k) > c_2$, it has no zeros. To write explicit expressions for the function $N(z,k)$, we divide the (β, z)-plane into domains (Fig. 2). Recalling

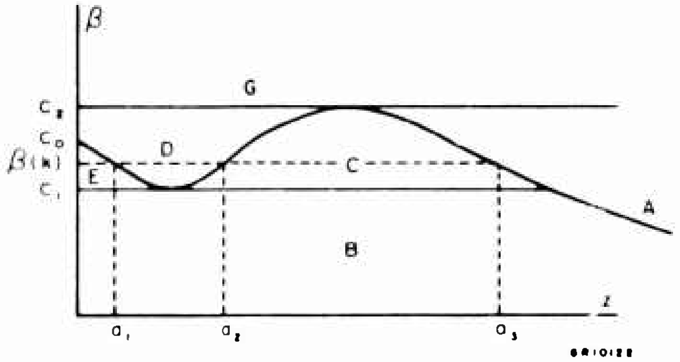

Fig. 2. Division of (β, z) plane into domains defined by the relative values of β and $c(z)$.

that for very large altitudes N is to represent upward travelling waves, the appropriate expression for this function in domains A, B, and C is

$$N = \left(\frac{u_3}{Q}\right)^{1/2} H_{1/3}^{(2)}(u_3) \quad , \tag{3·08}$$

$$u_3 = \int_{a_3}^{z} Q\, dz \quad . \tag{3·09}$$

Since $H_{1/3}^{(2)}$ is a multiple valued function of u, it is necessary to consider the way in which the phase angles to be assigned to the u's vary between the different domains. As indicated by Eq. (3·01), arg Q = 0 in domains A, D, and G, while in domains B, C, and E, arg Q = $-\pi/2$. In the neighborhood of a_3, Q is proportional to $(z - a_3)^{1/2}$, so that u_3 is proportional to $(z - a_3)^{3/2}$, or Q^3; thus arg u_3 = 3 arg Q. Near a_2, Q is proportional to $(a_2 - z)^{1/2}$ and u_2 is proportional to $-Q^3$, so that arg u_2 = 3 arg Q + π. Near a_1 we again have arg u_1 = 3 arg Q. The values to be assigned to arg u in the various domains then are given by the following scheme:

Domain	*arg Q*	*arg u_1*	*arg u_2*	*arg u_3*
A	0	-	-	0
B	$-\pi/2$	-	-	$-3\pi/2$
C	$-\pi/2$	-	$-\pi/2$	$-3\pi/2$
D	0	0	π	-
E	$-\pi/2$	$-3\pi/2$	-	-

In domain G, Q has no zeros and we may choose a arbitrarily. In this case N is not represented by the same combination of Bessel functions on opposite sides of a, but the leading term in its asymptotic expansion is the same on both sides. For continuity with domain A we set a = z_2 in domain G.

The leading terms in the asymptotic expansions of $H_{1/3}^{(1)}(u)$ and $H_{1/3}^{(2)}(u)$ for the values of arg u given in the above table are as follows[7]:

arg u = $-\pi/2$, 0, or π

$$H_{1/3}^{(1)}(u) \rightarrow (2/\pi u)^{1/2} \exp[i(u - 5\pi/12)] \tag{3·10}$$

[7] G. N. Watson, *Bessel Functions*, 2nd ed.; Cambridge University Press; pp 75-201.

$\arg u = -3\pi/2$

$$H^{(1)}_{1/3}(u) \longrightarrow (2/\pi u)^{1/2} \{\exp[i(u - 5\pi/12)] + \exp[-i(u + 11\pi/12)]\} \qquad (3\cdot11)$$

$\arg u = -3\pi/2, \ -\pi/2, \ \text{or} \ 0$

$$H^{(2)}_{1/3}(u) \longrightarrow (2/\pi u)^{1/2} \exp[-i(u - 5\pi/12)] \qquad (3\cdot12)$$

$\arg u = \pi$

$$H^{(2)}_{1/3}(u) \longrightarrow (2/\pi u)^{1/2} \{\exp[i(u + 11\pi/12) + \exp[-i(u - 5\pi/12)]\} \quad . \qquad (3\cdot13)$$

We also define the following positive real quantities:

$$\left.\begin{aligned}
y_1(z) &= -\frac{iu_1}{k} = \int_z^{a_1} \sqrt{1 - [\beta(k)/c(z)]^2}\, dz & 0 < z < a_1 \\
y_2(z) &= \frac{iu_2}{k} = \int_{a_2}^{z} \sqrt{1 - [\beta(k)/c(z)]^2}\, dz & a_2 < z < a_3 \\
y_3(z) &= -\frac{iu_3}{k} = \int_z^{a_3} \sqrt{1 - [\beta(k)/c(z)]^2}\, dz & a_2 < z < a_3
\end{aligned}\right\} \qquad (3\cdot14)$$

$$\left.\begin{aligned}
x_1(z) &= \frac{u_1}{k} = \int_{a_1}^{z} \sqrt{[\beta(k)/c(z)]^2 - 1}\, dz & a_1 < z < a_2 \\
x_2(z) &= -\frac{u_2}{k} = \int_z^{a_2} \sqrt{[\beta(k)/c(z)]^2 - 1}\, dz & a_1 < z < a_2 \\
x_3(z) &= \frac{u_3}{k} = \int_{a_3}^{z} \sqrt{[\beta(k)/c(z)]^2 - 1}\, dz & a_3 < z
\end{aligned}\right\} \qquad (3\cdot15)$$

By using the connection relations (3·07) with the asymptotic expressions (3·10) to (3·13), we find the following expressions for the function N(z,k) in the various domains:

In A,

$$N = \left(\frac{u_3}{Q}\right)^{1/2} H^{(2)}_{1/3}(u_3) \longrightarrow \left(\frac{2}{\pi Q}\right)^{1/2} \exp\left\{-i\left(kx_3(z) - \frac{5\pi}{12}\right)\right\} . \qquad (3\cdot16)$$

In B,

$$N = \left(\frac{u_3}{Q}\right)^{1/2} H^{(2)}_{1/3}(u_3) \longrightarrow \left(\frac{2}{\pi Q}\right)^{1/2} \exp\left(ky_3(z) + \frac{5\pi i}{12}\right) . \qquad (3\cdot17)$$

In C,

$$N = \left(\frac{u_2}{Q}\right)^{1/2} \exp\,[ky_2(a_3)]H^{(2)}_{1/3}(u_2) \longrightarrow \left(\frac{2}{\pi Q}\right)^{1/2} \exp\left(-ky_2(z) + ky_2(a_3) + \frac{5\pi i}{12}\right) \qquad (3\cdot18)$$

In D,

$$N = \left(\frac{u_2}{Q}\right)^{1/2} \exp[ky_2(a_3)]H^{(2)}_{1/3}(u_2) \longrightarrow 2\left(\frac{2}{\pi Q}\right)^{1/2} \exp\left(ky_2(a_3) + \frac{2\pi i}{3}\right)\cos\left(kx_2(z) - \frac{\pi}{4}\right) . \qquad (3\cdot19)$$

In E,

$$N = \left(\frac{u_1}{Q}\right)^{1/2} \exp[ky_2(a_3)]\left\{\exp\left(-ikx_2(a_1) + \frac{4\pi i}{3}\right)H^{(1)}_{1/3}(u_1) + \exp[ikx_2(a_1)]H^{(2)}_{1/3}(u_1)\right\}$$

$$\longrightarrow \left(\frac{2}{\pi Q}\right)^{1/2} \exp\left(ky_2(a_3) + \frac{2\pi i}{3}\right) \left\{\exp\left(-ky_1(z) - ikx_2(a_1) + \frac{\pi i}{4}\right) + \right. \qquad (3\cdot20)$$

$$\left. 2\cos kx_2(a_1)\exp\left(ky_1(z) - \frac{\pi i}{4}\right)\right\}$$

In G,

$$N \longrightarrow \left(\frac{2}{\pi Q}\right)^{1/2} \exp\left(-i(u - \frac{5\pi}{12})\right)$$

$$(3\cdot21)$$

$$u = \int_{z_2}^{z} Q\,dz .$$

With these expressions for N, there are discontinuities in F(z,k) with respect to variations of k at $k = \omega/c_1$ and ω/c_2. The transformation of the path of integration

in the integral expression for ϕ leads, in addition to the contributions from the residues given by Eq. (2·10), to integrals to and from $\omega/c_2 - i\infty$, $\omega/c_1 - i\infty$, and $\omega/c(z) - i\infty$ around the cuts in the k-plane shown in Fig. 3. There is no discontinuity

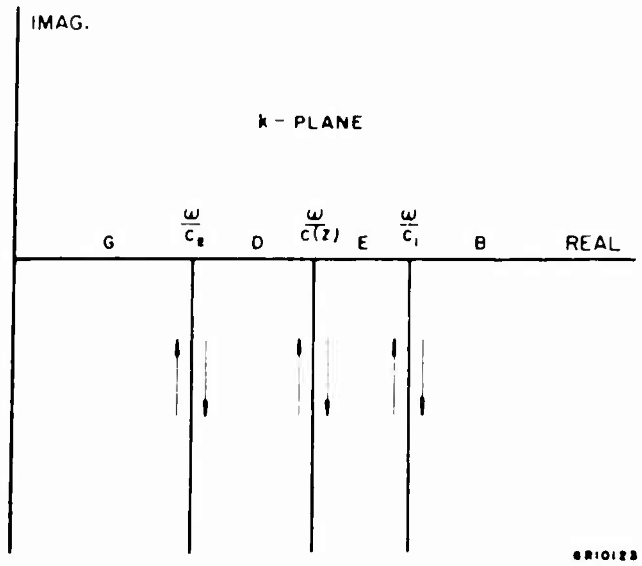

Fig. 3. Paths of integration around cuts in complex k-plane.

with respect to k at the cut starting at $\omega/c(z)$ on the real axis, so that the integral around this cut vanishes. The integral around the cut starting at ω/c_1 does not vanish because of the discontinuity in F between domains B and E, but it may be shown that this integral is of order r^{-1} as compared to the residue terms which are of order $r^{-1/2}$. This integral is therefore negligible when r is large. Similarly, the integral around the cut starting at ω/c_2 does not vanish. It will appear presently that this integral is to be interpreted as representing the effect of rays that leave the source at angles steep enough so that they are not refracted back below the level z_2. This integral, therefore, is negligible for large values of r when $z < z_2$, which is the only case which concerns us, but it would not be negligible if we wished to compute values of ϕ for $z > z_2$.

4. THE ROOTS OF $N'(0,k)$

We have

$$\frac{d}{dz}\left[\left(\frac{u}{Q}\right)^{1/2} H_{1/3}(u)\right] = \left[\frac{1}{6}\left(\frac{Q}{u}\right)^{1/2} - \frac{1}{2}\left(\frac{u}{Q}\right)^{1/2}\frac{Q'}{Q}\right] H_{1/3}(u) + (uQ)^{1/2} H_{-2/3}(u) \quad , \tag{4·01}$$

where H stands for either of the Bessel functions $H^{(1)}$ or $H^{(2)}$. For a given value of $\beta(k)$ the expression in brackets in Eq. (4·01) is independent of ω, while $(uQ)^{1/2}$ is proportional to ω; hence we may write

$$\frac{d}{dz}\left[\left(\frac{u}{Q}\right)^{1/2} H_{1/3}\right] \simeq (uQ)^{1/2} H_{-2/3}(u) \tag{4·02}$$

to the same approximation as that involved in writing N in terms of $H_{1/3}$. Since the leading term in the asymptotic expansion of $H_{-2/3}$ may be obtained from that for $H_{1/3}$ by multiplying the e^{iu} term by $e^{\pi i/2}$ and the e^{-iu} term by $e^{-\pi i/2}$ for any value of arg u, the asymptotic expressions for $N'(0,k)$ in the various domains are derived readily from those for N by inspection.

In B,

$$N'(0,k) = [u_3(0)\ Q(0)]^{1/2} H^{(2)}_{-2/3}[u_3(0)] \longrightarrow \left[\frac{2Q(0)}{\pi}\right]^{1/2} \exp\left[ky_3(0) - \frac{\pi i}{12}\right] \ . \tag{4·03}$$

In E,

$$N'(0,k) = [u_1(0)\ Q(0)]^{1/2} \exp[ky_2(a_3)] \left\{\exp\left[-ikx_2(a_1) + \frac{4\pi i}{3}\right]H^{(1)}_{-2/3}[u_1(0)] + \exp[ikx_2(a_1)]H^{(2)}_{-2/3}[u_1(0)]\right\}$$

$$\longrightarrow \left[\frac{2Q(0)}{\pi}\right]^{1/2} \exp\left[ky_2(a_3) + \frac{2\pi i}{3}\right] \left\{\exp\left[-ky_1(0) - ikx_2(a_1) + \frac{3\pi i}{4}\right] + \right.$$

$$\left. 2\cos kx_2(a_1) \exp\left[ky_1(0) - \frac{3\pi i}{4}\right]\right\} \ . \tag{4·04}$$

In D,

$$N'(0,k) = [u_2(0)\ Q(0)]^{1/2} H^{(2)}_{-2/3}[u_2(0)] \longrightarrow 2\left[\frac{2Q(0)}{\pi}\right]^{1/2} \exp\left[ky_2(a_3) + \frac{2\pi i}{3}\right] \cos\left[kx_2(0) - \frac{3\pi}{4}\right] \ . \tag{4·05}$$

In G,

$$N'(0,k) \longrightarrow \left[\frac{2Q(0)}{\pi}\right]^{1/2} \exp\left\{-i\left[u(0) + \frac{\pi}{12}\right]\right\} \tag{4·06}$$

From Eq. (4·03) we see that $N'(0,k)$ has no zeros for $\beta(k) < c_1$, so that the series (2·10) begins with values of k_n such that $\beta(k_n) > c_1$. From Eq. (4·04) it appears that there are no real roots in the range $c_1 < |\beta(k_n)| < c_0$. It should be noted, however, that the asymptotic expansions of the Bessel functions are valid only when the quantities kx and ky are large. Consequently $\exp[-ky_1(0)]$ is very small compared to

$\exp[ky_1(0)]$ and the values of k_n in this range will be given very nearly by $\cos k_n x_2(a_1) = 0$ or

$$k_n x_2(a_1) \simeq (n - \frac{1}{2})\pi \quad , \tag{4·07}$$

where n is a positive integer. From Eq. (4·05) the roots in the range $c_0 < \beta(k) < c_2$ are given by

$$k_n x_2(0) = (n - \frac{3}{4})\pi \quad , \tag{4·08}$$

and from Eq. (4·06) there are no roots for $\beta(k) > c_2$.

Equations (4·07) and (4·08) for k_n, being derived from the asymptotic expansions, are not valid for small values of n, although they are correct to three significant figures or better for $n \geq 4$. Explicit expressions for the lower-order roots in domain E cannot be derived readily, but in domain D they may be computed from the equation

$$H^{(2)}_{-2/3}[u_2(0)] = \left(\frac{2}{\sqrt{3}}\right) \varepsilon^{\pi i/6}\left[J_{2/3}[kx_2(0)] - J_{-2/3}[kx_2(0)]\right] \quad . \tag{4·09}$$

The first five roots of this expression are $k_n x_2(0) = 0.6855$, 3.9028, 7.0549, 10.2007, and 13.3445. For frequencies higher than about 0.2 cycle the sum (Eq. 2·10)) for ϕ includes a large number of high-order modes, and the contribution from the few low-order modes is comparatively small. For very much lower frequencies the only normal mode solutions that exist are those of low order. At such frequencies, however, the representation of the function $N(z,k)$ in terms of Bessel functions becomes a poor approximation. The present theory therefore may be considered valid only for frequencies sufficiently high for the major contribution to ϕ to come from modes of order $n > 4$. For these modes the asymptotic forms of the Bessel functions provide a sufficiently accurate representation, and these forms accordingly will be used in all subsequent development.

5. ASYMPTOTIC APPROXIMATION FOR ϕ

We now have explicit expressions for all the quantities appearing in Eq. (2·10) except the factor $[\partial N'(0,k)/\partial k]_{k=k_n}$.

From the definition (3·15),

$$\frac{d}{dk}\left[kx_2(a_1)\right] = -k\int_{a_1}^{a_2} \frac{dz}{Q} \qquad c_1 < \beta(k) < c_0 \quad . \tag{5·01}$$

For $\beta(k) > c_0$, a_1 is replaced by 0 in this expression.

From Eqs. (4·04), (4·05), and (5·01) with the condition $N'(0,k_n) = 0$, we find

$$\left.\frac{\partial N'(0,k)}{\partial k}\right|_{k=k_n} \to 2k_n(-1)^{n-1}\exp\left[k_n y_2(a_3) + k_n y_1(0) - \frac{\pi i}{12}\right]\left[\frac{2Q_n(0)}{\pi}\right]^{1/2}\int_{a_1}^{a_2}\frac{dz}{Q_n} \qquad (5\cdot02)$$

for $c_1 < \beta(k_n) < c_0$, and

$$\left.\frac{\partial N'(0,k)}{\partial k}\right|_{k=k_n} \to 2k_n(-1)^{n}\exp\left[k_n y_2(a_3) + \frac{2\pi i}{3}\right]\left[\frac{2Q_n(0)}{\pi}\right]^{1/2}\int_{0}^{a_2}\frac{dz}{Q_n} \qquad (5\cdot03)$$

for $c_0 < \beta(k_n) < c_2$. From Eq. (3·20) combined with (5·02) and from (3·10) with (5·03) we have

$$N(0,k_n)\left[\frac{\partial N'(0,k)}{\partial k}\right]_{k=k_n} \to \frac{8k_n}{\pi}\exp\left[2k_n y_2(a_3) + \frac{\pi i}{3}\right]\int_{a_1}^{a_2}\frac{dz}{Q_n} \qquad (5\cdot04)$$

for $c_1 < \beta(k_n) < c_0$ and the same expression with 0 replacing a_1 for $c_0 < \beta(k_n) < c_2$.

We shall assume the source to be at an altitude less than that of the velocity minimum, that is, $h < z_1$, so that $N(h,k_n)$ is given by Eqs. (3·19) and (3·20).

$$N(h,k_n) \to \begin{cases} 2\left[\dfrac{2}{\pi Q_n(h)}\right]^{1/2}(-1)^{n+1}\exp\left[k_n y_2(a_3) - k_n y_1(0) + \dfrac{5\pi i}{12}\right]\cosh\left[k_n\{y_1(h) - y_1(0)\}\right] & c_1 < \beta(k_n) < c(h) \\ 2\left[\dfrac{2}{\pi Q_n(h)}\right]^{1/2}\exp\left[k_n y_2(a_3) + \dfrac{2\pi i}{3}\right]\cos\left[k_n x_2(h) - \dfrac{\pi}{4}\right] & c(h) < \beta(k_n) < c_2 \end{cases} \qquad (5\cdot05)$$

Substituting Eqs. (5·04) and (5·05) in Eq. (2·10) and using the asymptotic form of $H_0^{(2)}(k_n r)$, we may write

$$\Phi_{res} = \Phi_1 + \Phi_2 + \Phi_3$$

where

$$\phi_1 = \pi A \exp[i(\omega t - \pi/6)] \sum_{c_1<\beta(k_n)<c(h)} (-1)^{n+1} \left\{\frac{1}{rk_nQ_n(h)}\right\}^{1/2} \frac{\exp[-ik_nr - k_ny_2(a_3) - k_ny_1(0)]}{\int_{a_1}^{a_2} dz/Q_n} \times$$

$$\cosh\left\{k_n[y_1(h) - y_1(0)]\right\} N(z,k_n) \quad , \qquad (5\cdot06)$$

$$\phi_2 = \pi A \exp[i(\omega t + \pi/12)] \sum_{c(h)<\beta(k_n)<c_0} \left\{\frac{1}{rk_nQ_n(h)}\right\}^{1/2} \frac{\exp[-ik_nr - k_ny_2(a_3)]}{\int_{a_1}^{a_2} dz/Q_n} \times$$

$$\cos[k_nx_2(h) - \pi/4)N(z,k_n)] \quad , \qquad (5\cdot07)$$

$$\phi_3 = \pi A \exp[i(\omega t + \pi/12)] \sum_{c_0<\beta(k_n)<c_2} \left\{\frac{1}{rk_nQ_n(h)}\right\}^{1/2} \frac{\exp[-ik_nr - k_ny_2(a_3)]}{\int_{0}^{a_2} dz/Q_n} \times$$

$$\cos[k_nx_2(h) - \pi/4]N(z,k_n) \quad . \qquad (5\cdot08)$$

Now each term of ϕ_1 contains the factors $\exp[-k_ny_2(a_3)]$ and $\exp[-k_ny_1(0)]$, both of which are very small. Except when $[z,\beta(k_n)]$ fails in domain A, the factor $\exp[-k_ny_2(a_3)]$ is cancelled by its inverse in $N(z,k_n)$, but the factor $\exp[-k_ny_1(0)]$ remains in all cases. Thus ϕ_1 is negligibly small for all values of z. ϕ_2 is likewise negligible because of the factor $\exp[-k_ny_2(a_3)]$ when $[z, \beta(k_n)]$ is in domain A; and all terms in ϕ_2 for which $[z,\beta(k_n)]$ falls in domain C are also negligible because of the factor $\exp[-k_ny_2(z)]$ which appears in $N(z,k_n)$ in this domain. When $[z,\beta(k_n)]$ falls in domain E, Eq. (3·20) gives

$$N(z,k_n) \longrightarrow 2\left\{\frac{2}{\pi Q_n(z)}\right\}^{1/2} (-1)^{n+1} \exp\left[k_ny_2(a_3) - k_ny_1(0) + \frac{5\pi i}{12}\right]\cosh\left[k_n\{y_1(z) - y_1(0)\}\right] \quad ,$$

and the corresponding terms of ϕ_2 are negligible because of the factor $\exp[-k_ny_1(0)]$. The surviving terms of ϕ_2, therefore, are those for which $[z,\beta(k_n)]$ lies in domain D.

For these we have from Eq. (3·19)

$$\phi_2 = 2A \exp[i(\omega t + 3\pi/4)] \sum_{c(h)<\beta(k_n)<c_0} \left[\frac{2\pi}{rk_nQ_m(h)Q_n(z)}\right]^{1/2} \frac{\exp(-ik_nr)}{\int_{a_1}^{a_2} dz/Q_n} \cos[k_nx_2(h) - \pi/4] \times$$

$$\cos[k_nx_2(z) - \pi/4]\,\delta[z, a_1(k_n), a_2(k_n)] \quad , \qquad (5\cdot09)$$

where $\delta[z,a_1(k_n),a_2(k_n)]$ is a factor that has the value 1 for $a_1(k_n) < z < a_2(k_n)$ and 0 for z outside this range. A similar argument leads to

$$\phi_3 = 2A \exp[i(\omega t + 3\pi/4)] \sum_{c_0<\beta(k_n)<c_2} \left[\frac{2\pi}{rk_nQ_n(h)Q_m(z)}\right]^{1/2} \frac{\exp(-ik_nr)}{\int_0^{a_2} dz/Q_n} \cos[k_nx_2(h) - \pi/4] \times$$

$$\cos[k_nx_2(z) - \pi/4]\,\delta[z, 0, a_2(k_n)] \quad . \qquad (5\cdot10)$$

In Eqs. (5·09) and (5·10) the values of k_n are to be determined by Eqs. (4·07) and (4·08), respectively.

6. GEOMETRICAL INTERPRETATION OF THE NORMAL MODES

It will be shown that the summations indicated in Eqs. (5·09) and (5·10) may be carried out by replacing the sums by equivalent integrals, which then may be evaluated approximately by the method of stationary phase. Before proceeding with this development, however, it will be instructive to consider the significance in terms of ray geometry of the functions $Q(z,k)$ and $x_2(z,k)$ that express the dependence of ϕ on z. Let $\theta(z)$ be the angle between a given geometrical ray and the horizontal. Then Snell's law is $c(z) \sec \theta(z) = \text{constant}$. Since the function $x_2(z)$ is defined only for values of $\beta(k) > c(z)$, $\beta(k)$ is a possible ray constant and defines a ray for which $\sec \theta(z,k) = \beta(k)/c(z)$. Equation (3·01) for this case may be written in the form

$$Q(z,k) = k\sqrt{\sec^2\theta(z,k) - 1} = k \tan \theta(z,k) \quad . \qquad (6\cdot01)$$

The zeros of $Q(z,k)$ are thus the altitudes at which the corresponding ray is horizontal. From the second of equations (3·15) we have

$$x_2(z) = \int_z^{a_2} \tan\theta(z,k)\,dz$$

and writing
$$\tan\theta = \frac{\beta(k)}{c(z)}\csc\theta - \cot\theta$$

this takes the form

$$x_2(z) = \beta(k)\int_z^{a_2} \frac{\csc\theta(z,k)}{c(z)}\,dz - \int_z^{a_2} \cot\theta(z,k)\,dz$$

The first integral in this expression is the travel time along the given ray from the altitude z to the apex of the ray at altitude a_2, and the second is the horizontal distance traversed (Fig. 4). Calling these quantities $t_2(z,k)$ and $r_2(z,k)$, respectively, we write

$$x_2(z,k) = \beta(k)t_2(z,k) - r_2(z,k) \quad . \tag{6·02}$$

Furthermore,

$$\int_{a_1}^{a_2} \frac{dz}{Q} = \frac{1}{k}\int_{a_1}^{a_2} \cot\theta\,dz = \frac{r_2(a_1,k)}{k} \quad . \tag{6·03}$$

The general character of the function $x_2(z,k)$ for the type of velocity altitude function assumed is shown in Fig. 5.

By writing the factor $\cos[kx_2(z) - \pi/4]$ in exponential form, it may be seen that each term in Eqs. (5·09) and (5·10) represents a superposition of a pair of progressive waves whose phases are given (apart from a constant term) by

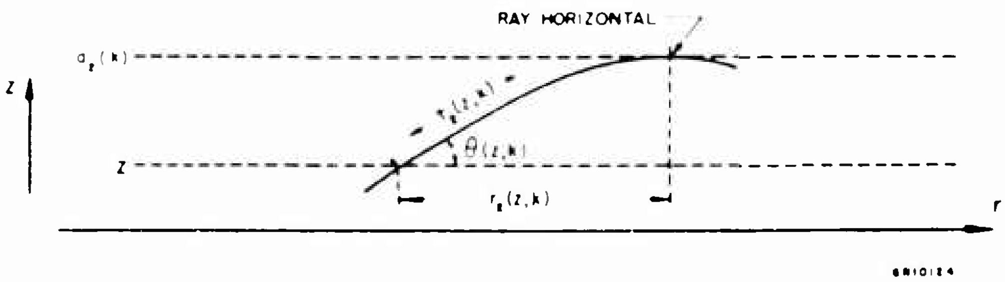

Fig. 4. Ray path in (r,z) plane.

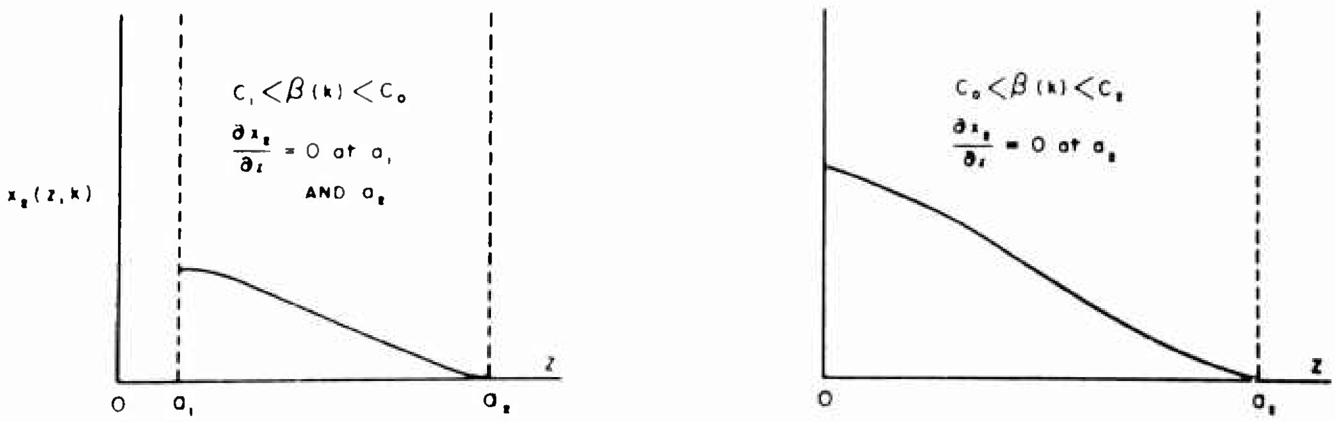

Fig. 5. General character of the function.

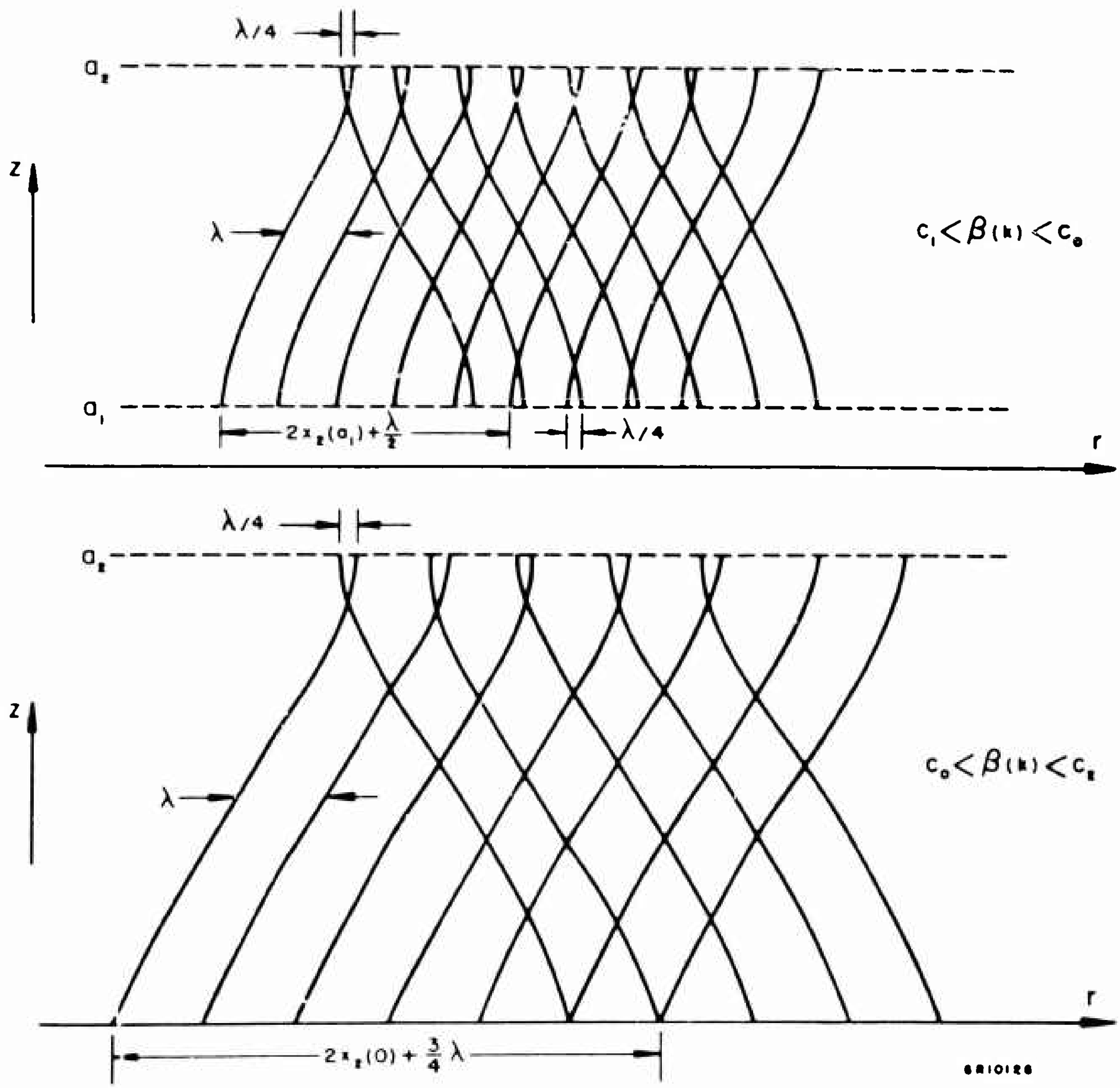

Fig. 6. Configuration of normal mode wave fronts.

$$\omega\left(t - \frac{r - x_2(z,k_n)}{\beta(k_n)}\right) - \frac{\pi}{4}$$

and

$$\omega\left(t - \frac{r + x_2(z,k_n)}{\beta(k_n)}\right) + \frac{\pi}{4} \quad .$$

Thus the parameter $\beta(k_n)$ is to be interpreted as the phase velocity of the n^{th} normal mode, and $k_n = 2\pi/\lambda_n$ where λ_n is the corresponding wavelength (measured horizontally). The configuration of the wave fronts, or loci of constant phase, is illustrated in Fig. 6. This configuration is such that the velocity of propagation of any portion of a wave front measured along the normal is c(z), while the velocity measured horizontally is the same for all values of z and is equal to $\beta(k_n)$.

The conditions (4·07) and (4·08) that determine the k's may be written in the form

$$2x_2(a_1) \;=\; \lambda_n(n - \tfrac{1}{2}) \qquad c(h) < \beta(k_n) < c_0 \tag{6·04}$$

$$2x_2(0) \;=\; \lambda_n(n - \tfrac{3}{4}) \qquad c_0 < \beta(k_n) < c_2 \quad . \tag{6·05}$$

These are the conditions which must be satisfied if each of the two sets of wave fronts is to be in phase with the reflected or refracted image of the other, with allowance for a quarter-wavelength phase shift when the wave front is reversed by refraction and no phase shift when the reversal is by reflection at the ground surface. This is exactly analogous to the condition for constructive interference between multiple internal reflections that determine the normal modes in the case of a medium made up of discrete layers.[5]

We also observe that the (asymptotic) vanishing of ϕ_1, which corresponds to the non-excitation of the normal modes for which $\beta(k_n) < c(h)$, is the expression in the normal mode representation of the fact that the corresponding rays do not pass through the level of the source. The interpretation of $\beta(k)$, for real values of k, as a quantity defining a family of rays in the (r,z)-plane indicates the physical meaning of the integrals around the cuts in the k-plane. The integral around the cut starting at ω/c_1 arose by transformation of the original path of integration along the real axis for $0 < \beta(k) \leq c_1$. The corresponding rays are confined to the portion of the (r,z)-plane for which z is greater than the altitude a_3 (Fig. 7). Such rays, therefore, do not contribute appreciably to the value of ϕ when either h or z is less than a_3, which is the case in which we are primarily interested. Similarly, the integral around the cut starting at ω/c_2 represents the contribution to ϕ of rays for which $\beta(k) \geq c_2$,

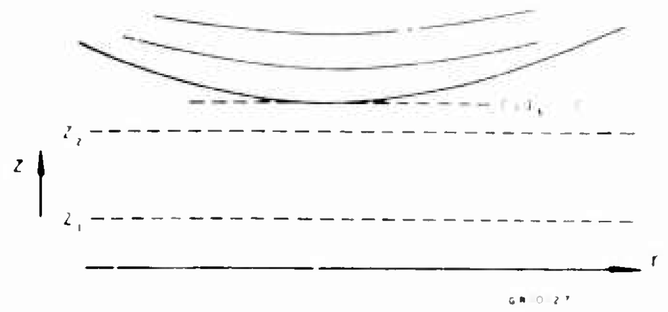

Fig. 7. Rays for which $0 < \beta(k) \leq c$.

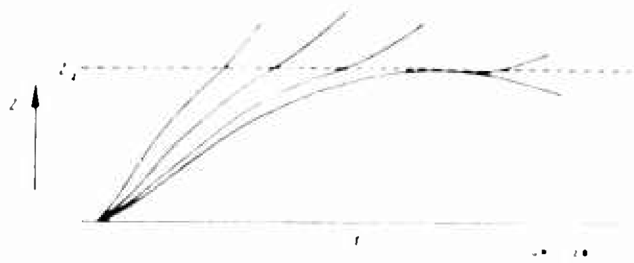

Fig. 8. Rays for which $\beta(k) \geq c_2$.

that is, of the type shown in Fig. 8. These rays, after once passing above the level z_2, thereafter do not return to lower elevations and thus do not contribute appreciably to ϕ when the point (r,z) lies to the right of the upward refracted branch of the critical ray that becomes horizontal at the altitude z_2. For such points, therefore, ϕ_{res} represents essentially the whole solution.

7. PHASE AND GROUP VELOCITIES OF THE NORMAL MODES

Since, for a given value of n, $\beta(k_n)$ is a function of frequency defined by Eqs. (4·07) or (4·08), the individual terms of Eqs. (5·09) and (5·10) represent dispersed waves. Therefore, it is of interest to compute the corresponding group velocities. The group velocity U_n corresponding to the phase velocity $\beta_n = \beta(k_n)$ is defined by

$$U_n = d\omega/dk_n \quad . \tag{7·01}$$

From Eq. (4·07) we have for n = constant

$$x_2(a_1)dk_n + k_n\, dx_2(a_1) = 0 \quad ,$$

or

$$\left\{x_2(a_1) + k_n\left(\frac{\partial x_2(a_1)}{\partial k_n}\right)_\omega\right\} dk_n + k_n\left(\frac{\partial x_2(a_1)}{\partial \omega}\right)_{k_n} d\omega = 0 \quad . \tag{7·02}$$

From the fact that

$$x_2(a_1) = \frac{1}{k_n}\int_{a_1}^{a_2} \sqrt{\left(\frac{\omega}{c(z)}\right)^2 - k_n^2}\, dz$$

and the fact that the integrand vanishes at a_1 and a_2, we have

$$\left(\frac{\partial x_2(a_1)}{\partial \omega}\right)_{k_n} = \frac{1}{k_n}\int_{a_1}^{a_2} \frac{dz}{c(z)\sqrt{1-\left(\frac{c(z)}{\beta(k_n)}\right)^2}} = \frac{1}{k_n}\int_{a_1}^{a_2} \frac{dz}{c(z)\sin\theta} = \frac{t_2(a_1)}{k_n} \qquad (7\cdot 03)$$

and

$$\left(\frac{\partial x_2(a_1)}{\partial k_n}\right)_{\omega} = -\frac{1}{k_n}\int_{a_1}^{a_2} \tan\theta \, dz - \frac{1}{k_n}\int_{a_1}^{a_2} \cot\theta \, dz = -\frac{x_2(a_1) + r_2(a_1)}{k_n} \qquad (7\cdot 04)$$

$$= -\frac{\beta_n t_2(a_1)}{k_n} \ ,$$

whence

$$U_n = \frac{r_2(a_1)}{t_2(a_1)} \ . \qquad (7\cdot 05)$$

Equation (4·08) leads to the same relations with a_1 replaced by 0. Thus the group velocity of the n^{th} mode is equal to the average horizontal velocity along the corresponding ray between its maximum and minimum altitudes.

A numerical example is plotted in Figs. 9(a) and 9(b) which show β_n and U_n as functions of frequency for the following velocity-altitude function:

Alt. (km)	*c (m/sec)*
0	344
16.5	286
32	312
50	380

Linear velocity gradients are assumed between the tabulated values, and the velocity is assumed to decrease monotonically above 50 km. The fact that this velocity function does not fulfill the condition of continuity of dc/dz is beside the point since the ray geometry would not be greatly altered if we supposed the discontinuities in gradient to be rounded off to satisfy the continuity condition. The phase velocity curves provide the basis for the statement previously made that, for frequencies higher than about 0.2 cycle, Φ includes a large number of high-order modes. For example, the number of terms in Φ_3 for any given frequency is equal to the number of values of β_n at that frequency falling between the limits c_0 and c_2. Thus at 0.2 cycle there are 12 terms, from the

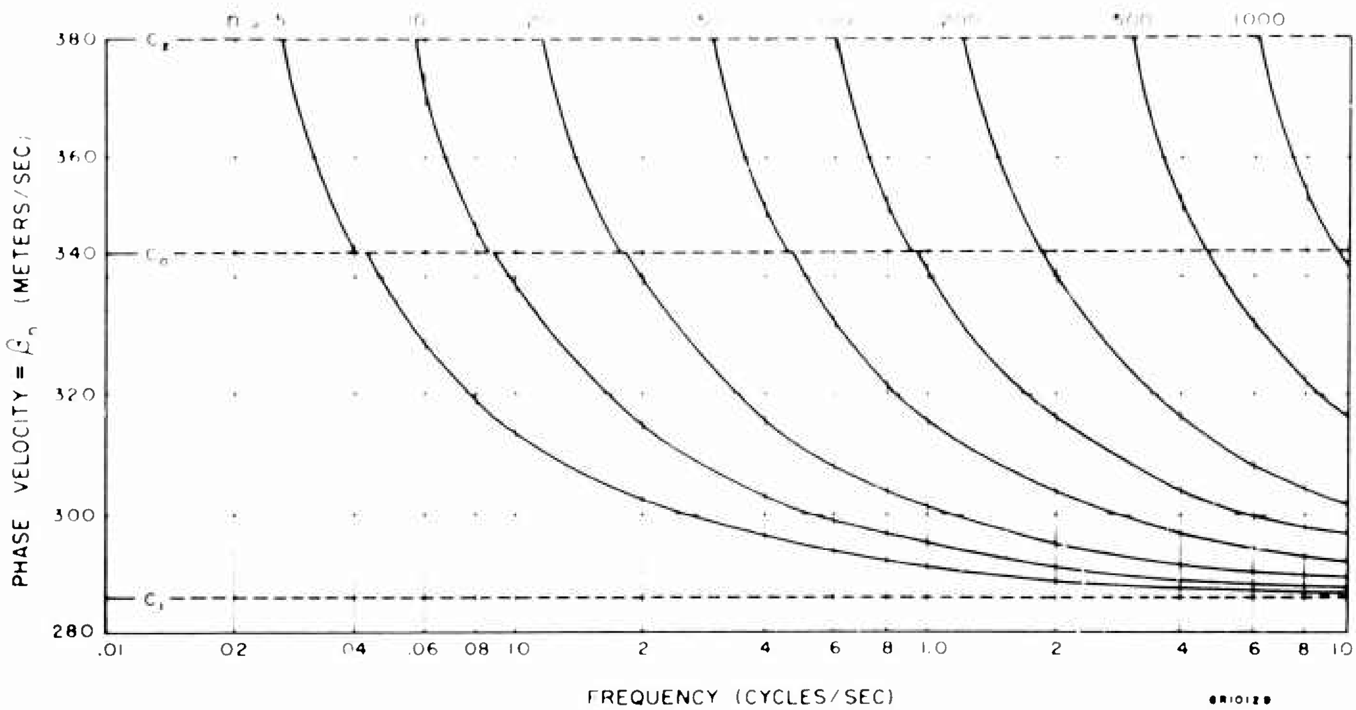

Fig. 9(a). Phase velocity of normal modes.

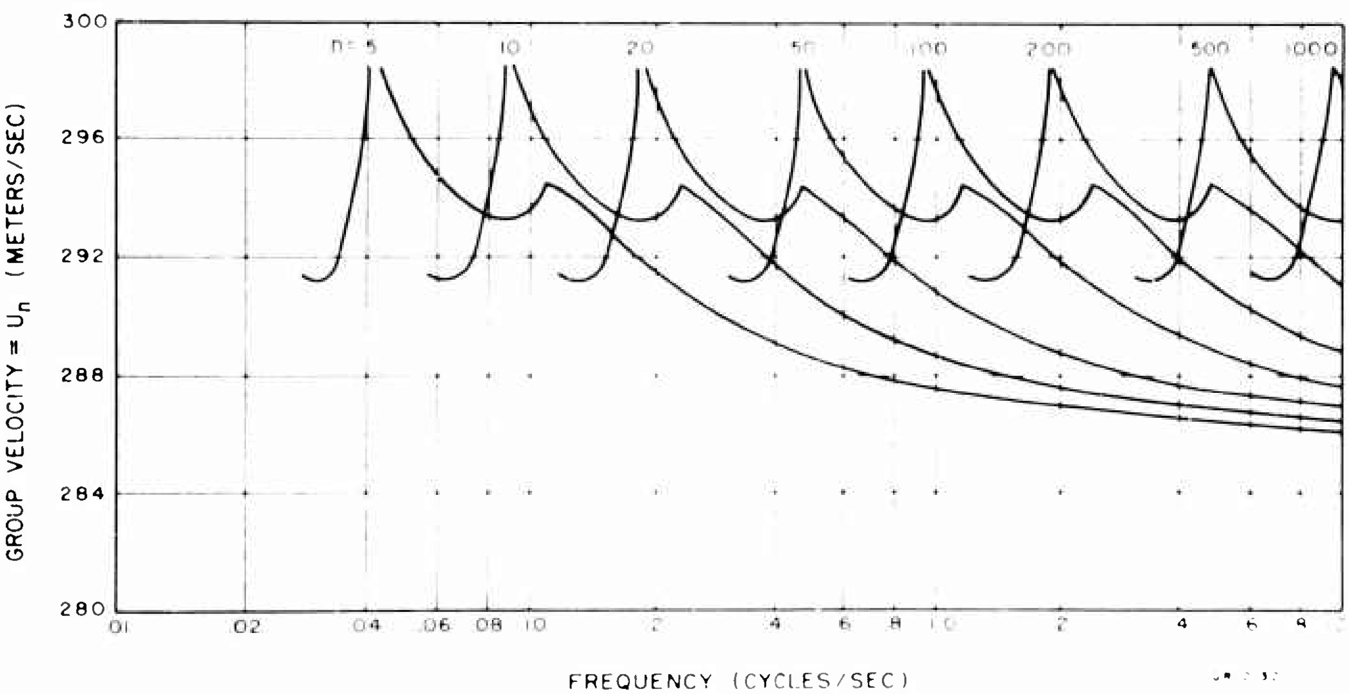

Fig. 9(b) Group velocity of normal modes.

22nd to the 33rd mode, inclusive; at 1.0 cycle there are 58 terms, from the 107th to the 164th mode; and at 5.0 cycles there are 288 terms, from the 532nd to the 819th mode.

8. INTEGRAL REPRESENTATION OF ϕ

Since the number of terms is so large, Eqs. (5·09) and (5·10) are not very useful as they stand. However, the fact that successive values of k_n differ only slightly suggests that the sums may be converted into equivalent integrals. Using the symbol Δ to denote the change in any quantity from the n^{th} to the $(n+1)^{th}$ mode, we have from either Eq. (4·07) or Eq. (4·08)

$$[1/\pi]\,\Delta\,[k_n x_2(a_1)] = \Delta n = 1 \quad .$$

But

$$\Delta\,[k_n x_2(a_1)] \simeq \Delta k_n \left\{ x_2(a_1) + k_n \left[\frac{\partial x_2(a_1)}{\partial k_n}\right]_\omega \right\} = -\, r_2(a_1)\,\Delta k_n$$

from Eq. (7·04). Hence

$$\frac{-r_2(a_1)}{\pi}\,\Delta k_n = 1 \quad . \tag{8·01}$$

Multiplying each term of Eqs. (5·09) and (5·10) by (8·01) and using Eqs. (6·01) and (6·03), we have

$$\phi_2 = 2A\,\exp[i(\omega t - \pi/4] \sum_{c(h)<\beta_n<c_0} \left[\frac{2}{\pi r k_n \tan\theta_n(z)\tan\theta_n(h)}\right]^{1/2} \exp(-ik_n r)\,\cos[k_n x_2(h) - \pi/4]\times$$

$$\cos[k_n x_2(z) - \pi/4]\,\delta(z,a_1,a_2)\Delta k_n \tag{8·02}$$

and

$$\phi_3 = 2A\,\exp[i(\omega t - \pi/4] \sum_{c_0<\beta_n<c_2} \left[\frac{2}{\pi r k_n \tan\theta_n(z)\tan\theta_n(h)}\right]^{1/2} \exp(-ik_n r)\,\cos[k_n x_2(h) - \pi/4]\times$$

$$\cos[k_n x_2(z) - \pi/4]\,\delta(z,0,a_2)\Delta k_n \quad . \tag{8·03}$$

Equation (8·01) shows that when $r_2(a_1)$ is a large number of wavelengths, Δk_n may be treated as a small quantity. However, before passing to the limit $\Delta k_n \rightarrow dk_n$ and writing these sums in the form of integrals over a continuous range of values of k_n, we must consider that in deriving Eqs. (5·09) and (5·10) terms depending upon $x_2(a_1)$ and $x_2(0)$ were eliminated by use of the relations (4·07) and (4·08). Before allowing k_n to assume continuous values we reintroduce the dependence upon $x_2(a_1)$ and $x_2(0)$ by noting that for any integer, s,

$$\exp\{-2is[k_n x_2(a_1) + \pi/2]\} = \exp(-2i\pi ns) = 1 \qquad c(h) < \beta_n < c_0$$

and

$$\exp\{-2is[k_n x_2(0) + 3\pi/4]\} = \exp(-2i\pi ns) = 1 \qquad c_0 < \beta_n < c_2 \quad .$$

These expressions, therefore, may be introduced as factors after the summation signs in Eqs. (8·02) and (8·03). The reason for the introduction of the arbitrary integer s will become apparent presently. Passing now to the limit $\Delta k_n \rightarrow dk$ and writing the cosine terms in exponential form, Eqs. (8·02) and (8·03) become

$$\Phi_2 = \frac{A}{2} \exp[i(\omega t - \pi/4 - s\pi)] \, (e^{-\pi i/2} I_1 + e^{\pi i/2} I_2 + I_3 + I_4) \tag{8·04}$$

and

$$\Phi_3 = \frac{A}{2} \exp[i(\omega t - \pi/4 - 3s\pi/2)] \, (e^{-\pi i/2} I_5 + e^{\pi i/2} I_6 + I_7 + I_8) \quad , \tag{8·05}$$

where

$$I_1 = \int_{k_h}^{k_0^+} f(k) \, \exp\{-ik[r - x_2(h) - x_2(z) + 2sx_2(a_1)]\} \, \delta(z, a_1, a_2) \, dk \quad , \tag{8·06}$$

$$I_2 = \int_{k_h}^{k_0^+} f(k) \, \exp\{-ik[r + x_2(h) + x_2(z) + 2sx_2(a_1)]\} \, \delta(z, a_1, a_2) \, dk \quad , \tag{8·07}$$

$$I_3 = \int_{k_h}^{k_0^+} f(k) \exp\{-ik[r - x_2(h) + x_2(z) + 2sx_2(a_1)]\} \delta(z, a_1, a_2)\, dk \quad , \qquad (8{\cdot}08)$$

$$I_4 = \int_{k_h}^{k_0^+} f(k) \exp\{-ik[r + x_2(h) - x_2(z) + 2sx_2(a_1)]\} \delta(z, a_1, a_2)\, dk \quad , \qquad (8{\cdot}09)$$

$$f(k) = \left[\frac{2}{\pi rk \tan\theta(z,k)\tan\theta(h,k)}\right]^{1/2} \quad . \qquad (8{\cdot}10)$$

The lower limit of integration, k_h, is the first value of k_n less than $\omega/c(h)$, and k_0^+ is the value of k_n next larger than ω/c_0. The integrals I_5 to I_8 are given by the same expressions with $a_1 = 0$ and the lower and upper limits of integration replaced by k_0^- and k_2, respectively, where k_0^- is the value of k_n next smaller than ω/c_0 and k_2 is the value next larger than ω/c_2.

9. APPROXIMATE EVALUATION OF THE INTEGRALS

In the approximate evaluation of integrals of the type (8·06) to (8·09), we must consider the criteria for the existence of stationary points and points of inflection of the phase of the exponential factors in the integrands. We let $X(z,h,k)$ stand for any one of the four quantities $2sx_2(a_1) \pm x_2(z) \pm x_2(h)$, so that the phase may be written in the form

$$\psi(r,z,h,k) = k[r + X(z,h,k)] \quad . \qquad (9{\cdot}01)$$

We also introduce the notation $R(z,h,k) = 2sr_2(a_1) \pm r_2(z) \pm r_2(h)$ and $T(z,h,k) = 2st_2(a_1) \pm t_2(z) \pm t_2(h)$. Then, using Eqs. (5·01) and (6·03) (which remain true if z or h is substituted for the lower limit of integration, a_1) we have

$$\frac{\partial\psi}{\partial k} = r - R(z,h,k) \quad , \tag{9·02}$$

$$\frac{\partial^2\psi}{\partial k^2} = -\frac{\partial R}{\partial k} \quad , \tag{9·03}$$

$$\frac{\partial^3\psi}{\partial k^3} = -\frac{\partial^2 R}{\partial k^2} \quad . \tag{9·04}$$

The condition for a point of stationary phase is then $r = R(z,h,k)$, which is to be regarded as an equation to determine a particular value (or set of values) of k for given values of r, z, and h. The geometrical significance of this condition is illustrated in Fig. 10, which shows a number of refracted cycles of the pair of rays specified by a particular value of k, say K. The radial distances to the points A_1, A_2,...., B_1, B_2, etc. at altitude z are given by the equations,

$$\begin{aligned}
r_A &= 2sr_2(a_1) + r_2(h) - r_2(z) & s &= 0, 1, 2, \ldots. \\
r_B &= 2sr_2(a_1) + r_2(h) + r_2(z) & s &= 0, 1, 2, \ldots. \\
r_C &= 2sr_2(a_1) - r_2(h) - r_2(z) & s &= 1, 2, 3, \ldots. \\
r_D &= 2sr_2(a_1) - r_2(h) + r_2(z) & s &= 1, 2, 3, \ldots. \quad ,
\end{aligned} \tag{9·05}$$

all of which are comprised in the general condition for the existence of a point of stationary phase for one or more of the integrals (8·06) to (8·09). Thus we see that whenever the point of observation (r,z) lies on a possible geometrical ray from the source,

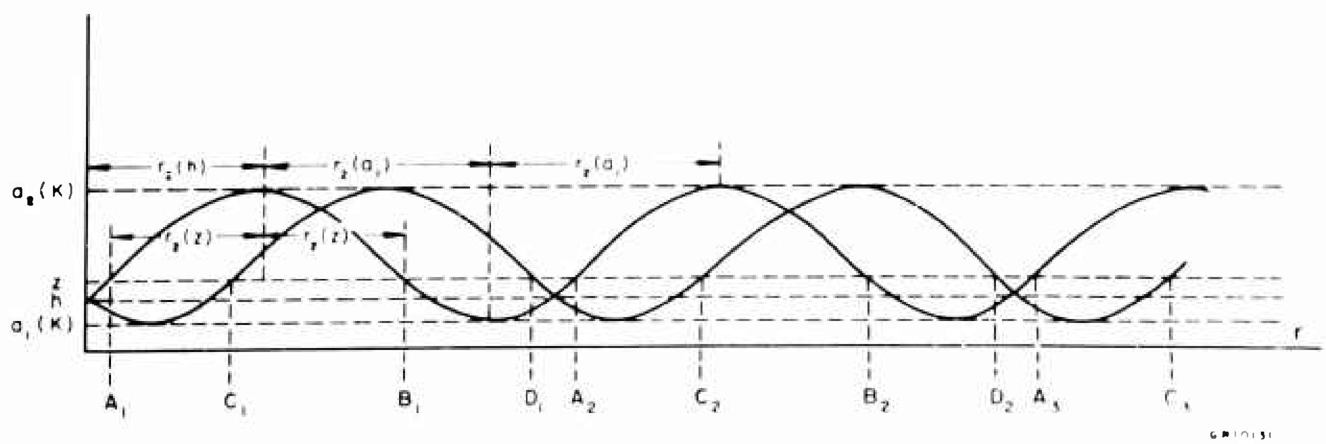

Fig. 10. Ray geometry and points of stationary phase.

at least one of the integrals I_1 to I_8 will have a point of stationary phase for some value of s. If (r,z) falls at a point where two rays having different values of K intersect, there will be two points of stationary phase; and if (r,z) lies in a geometrical shadow zone, the equation $r = R(z,h,k)$ will have no solution for real values of k and none of the integrals will have points of stationary phase. The integer s is now seen to represent the number of times the given ray passes through its minimum altitude (or is reflected at the ground surface) in reaching the point (r,z).

Approximate evaluations[8] of the general integral appropriate under various circumstances are as follows:

9.1 CASE I

(r,z) does not lie in a shadow zone and is not a focal point, so that $r = R(K)$, but $\partial R/\partial K \neq 0$.

$$I \simeq \left(-\frac{\pi}{\partial R/\partial K}\right)^{1/2} f(K) \exp\{-iK[r + X(K)]\} \int_{v_a}^{v_b} \exp[-i\pi v^2/2]\,dv \quad , \qquad (9{\cdot}06)$$

where

$$v_b = +\left\{\frac{2}{\pi}\left[K[r + X(K)] - k_b[r + X(k_b)]\right]\right\}^{1/2} \qquad (9{\cdot}07)$$

$$v_a = -\left\{\frac{2}{\pi}\left[K[r + X(K)] - k_a[r + X(k_a)]\right]\right\}^{1/2} \quad . \qquad (9{\cdot}08)$$

In the integrals I_1 to I_4, $k_b = k_0^+$, $k_a = k_h$, and in the integrals I_5 to I_8, $k_b = k_2$, $k_a = k_0^-$.

If the approximate evaluation of I represented by Eq. (9·06) is carried further, the next term is

$$i\left\{\frac{f(k_b) - f(K)}{r - R(k_b)} \exp\{-ik_b[r + X(k_b)]\} - \frac{f(k_a) - f(K)}{r - R(k_a)} \exp\{-ik_a[r + X(k_a)]\}\right\} \quad .$$

[8] See, for example, the treatment of the method of stationary phase given by C. Eckart, "Approximate Solutions of One Dimensional Wave Equations," *Rev. Mod. Phys.*, vol 20, p 399; 1948.

This term should be small if Eq. (9·06) is to be used as it stands. For a given value of $\beta(k)$, $f(k)$ and $\partial R/\partial k$ are proportional to $\omega^{-1/2}$, so that, apart from the fluctuations due to the presence of the Fresnel integral, the absolute value of the first term is independent of the frequency and the second is proportional to $\omega^{-1/2}$. In addition to this explicit dependence on the frequency, however, there is an implicit dependence due to the fact that, with increasing frequency, k_h approaches $\omega/c(h)$ so that $\tan\theta(h,k_h)$ approaches zero and $f(k)_h$ becomes very large. Therefore, we have no assurance that the second term will be small for any frequency. This convergence difficulty arises from the fact that the asymptotic approximations used for the function $N(z,\kappa)$ are not valid for values of z and k such that $\tan\theta$ is very small. If it were possible to obtain exact solutions of the differential equation, the convergence difficulty presumably would not occur. Therefore, we take Eq. (9·06) as it stands, without any correction term, as representing the best approximation obtainable by the present method. As justification for this procedure we shall show that, except for the diffraction effects represented by the Fresnel integral, Eq. (9·06) leads to exactly the same expression for the energy flux density that one derives on the basis of ray geometry.

When v_b and v_a are large in absolute value, the Fresnel integral approaches the value $(2i)^{1/2}$ so that

$$I \longrightarrow 2\left\{irK\frac{\partial R}{\partial K}\tan\theta(z,K)\tan\theta(h,K)\right\}^{-1/2}\exp\{-iK[r+X(K)]\} \quad . \qquad (9\cdot09)$$

Calling $\phi(K)$ the contribution to ϕ that arises from any particular ray through the point (r,z), we have

$$\mathrm{Amp}\,\phi(K) = A\left\{\left|RK\frac{\partial R}{\partial K}\tan\theta(z,K)\tan\theta(h,K)\right|\right\}^{-1/2} \qquad (9\cdot10)$$

In computing the particle velocity from ϕ we may treat $\mathrm{Amp}\,\phi$ as constant and differentiate only the exponential phase factor, since $\mathrm{Amp}\,\phi$ is, by comparison, a slowly varying quantity. By use of Eq.(1·07), the absolute value of the energy flux density is then

$$\begin{aligned} |J(K)| &= \frac{\rho(h)}{c(z)}[\omega\,\mathrm{Amp}\,\phi(K)]^2 \\ &= \frac{\rho(h)\omega^2A^2}{c(z)}\left\{\left|RK\frac{\partial R}{\partial K}\tan\theta(z,K)\tan\theta(h,K)\right|\right\}^{-1} \quad . \end{aligned} \qquad (9\cdot11)$$

From the viewpoint of ray geometry, we consider[9] the energy flux emitted in the solid angle between the rays that leave the source at angles θ (h,K) and θ (h,K) + dθ (Fig. 11). If E is the total rate of energy emission at the source, we have

$$dE = \frac{E}{2} \cos \theta(h, K)\, d\theta = \frac{E}{2}\, d\left[\sin \theta(h,K)\right] .$$

The area of wave front included between these rays at the point [z,R(K)] is

$$dS = -2\pi R \sin \theta(z,K)\, dR .$$

Hence

$$|J| = \frac{dE}{dS} = E\left\{4\pi R \sin \theta(z, K) \frac{dR}{d[\sin \theta(h, K)]}\right\}^{-1} \qquad (9{\cdot}12)$$

Now

$$\frac{dR}{d\left[\sin \theta(h, K)\right]} = \frac{\partial R}{\partial K}\left\{\frac{\partial\left[\sin \theta(h, K)\right]}{\partial K}\right\}^{-1} ,$$

and using

$$\sin \theta(h, K) = \left\{1 - K^2 c^2(h)/\omega^2\right\}^{1/2}$$

with

$$\cos \theta(z, K)/\cos \theta(h,K) = c(z)/c(h) ,$$

Eq. (9·12) may be written in the form

$$|J| = \frac{E\, c(h)}{4\pi c(z)}\left\{RK \frac{\partial R}{\partial K} \tan \theta(z,K) \tan \theta(h,K)\right\}^{-1} \qquad (9{\cdot}13)$$

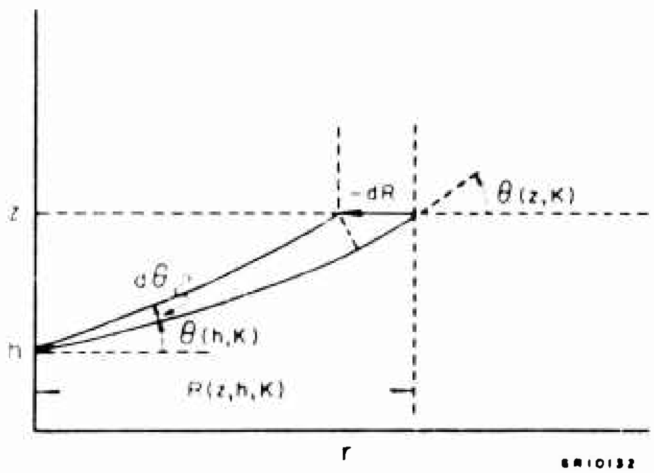

Fig. 11. Ray diagram near source.

From the limiting form approached by ϕ in the neighborhood of the point source, E and A are connected by the relation

$$E = \frac{4\pi\rho(h)\,\omega^2 A^2}{c(h)} , \qquad (9{\cdot}14)$$

which establishes the identity of Eqs. (9·13) and (9·11). Equation (9·06) thus has been shown to be completely equivalent to geometric ray theory in the high-frequency

[9] This treatment follows a similar derivation by B. Gutenberg, in "Propagation of Sound Waves in the Atmosphere," *J. Acoust. Soc. Am.*, vol 14, p 151, 1942. It is repeated here since the identity between Gutenberg's equation and Eq. (9·11) is not obvious without some changes in notation.

limit. The above argument also shows that Eqs. (1·05) and (1·06) conform to the requirements of energy conservation.

9.2 CASE II

(r, z) lies in a shadow zone where the boundary of the shadow zone is formed by a single ray (i.e., the boundary is not a caustic).

Let K be the parameter specifying the bounding ray. Then

$$I \simeq f(K)\int_{k_a}^{k_b} \exp\{-i[a + b(k-K) + c(k-K)^2]\}\,dk \quad ,$$

where $a = K[r + X(K)]$, $b = r - R(K)$, $c = -\frac{1}{2}\,\partial R/\partial K$. Substitution of

$$v = (2/\pi c)^{1/2}\,[c(k-K) + b/2]$$

as the variable of integration leads to

$$I \simeq \left[-\frac{\pi}{\partial R/\partial K}\right]^{1/2} f(K)\,\exp\left[-i\left\{K[r + X(K)] + \frac{[r - R(K)]^2}{2\,\partial R/\partial K}\right\}\right]\int_{v_a}^{v_b} \exp(-i\pi v^2/2)\,dv \quad , \tag{9·15}$$

where

$$v_a = \frac{r - R(K) - (k_a - K)\,\partial R/\partial K}{\sqrt{-\pi\,\partial R/\partial K}} \quad . \tag{9·16}$$

and

$$v_b = \frac{r - R(K) - (k_b - K)\,\partial R/\partial K}{\sqrt{-\pi\,\partial R/\partial K}} \quad . \tag{9·17}$$

9.3 CASE III

(r, z) lies near a point such that $\partial R/\partial k = 0$ is satisfied for some value of $k = K$. The appropriate expression for I in this case is

$$I \simeq \left[\frac{6}{d^2R/dK^2}\right]^{1/3} f(K)\,\exp\{-iK[r + X(K)]\}\int_{v_a}^{v_b} \exp[i(wv + v^3)]\,dv \quad , \tag{9·18}$$

where

$$w = [R(K) - r]\left[\frac{6}{\partial^2 R/\partial K^2}\right]^{1/3}, \tag{9·19}$$

$$v_a = \left[\frac{\partial^2 R/\partial K^2}{6}\right]^{1/3}(k_a - K), \tag{9·20}$$

$$v_b = \left[\frac{\partial^2 R/\partial K^2}{6}\right]^{1/3}(k_b - K). \tag{9·21}$$

When v_b and $-v_a$ are large in absolute magnitude, the integral in Eq. (9·18) approaches the Airy integral (except for a constant factor),

$$\int_{-\infty}^{+\infty} \exp[i(wv + v^3)]\, dv = \left\{\begin{array}{ll} \frac{2\pi}{3}\sqrt{\frac{|w|}{3}}\left[I_{-1/3}\left(\frac{2|w|}{3}\sqrt{\frac{|w|}{3}}\right) - I_{1/3}\left(\frac{2|w|}{3}\sqrt{\frac{|w|}{3}}\right)\right] & w > 0 \\ \frac{2\pi}{3}\sqrt{\frac{|w|}{3}}\left[J_{-1/3}\left(\frac{2|w|}{3}\sqrt{\frac{|w|}{3}}\right) - J_{1/3}\left(\frac{2|w|}{3}\sqrt{\frac{|w|}{3}}\right)\right] & w < 0 \end{array}\right\} \tag{9·22}$$

When (r,z) lies in a shadow zone, $w > 0$ and the Airy integral decreases monotonically with increasing $|w|$. When (r,z) is in a geometrical wave zone, $w < 0$ and the integral oscillates with slowly decreasing amplitude with increasing $|w|$. When $w = 0$ the integral has the value $2\pi/3\Gamma(2/3) = 1.5466$.

10. DEPENDENCE OF DIFFRACTION EFFECTS ON RANGE AND FREQUENCY

To illustrate the way in which the geometrically determined amplitude is modified by diffraction we consider the case in which the source and point of reception are at the ground surface, $z = h = 0$. For this case $\phi_2 = 0$ and in ϕ_3

$$I_5 = \int_{k_0^-}^{k_2} f(k)\; \exp\{-ik[r + 2(s-1)\, x_2(0)]\}\, dk, \tag{10·01}$$

$$I_6 = \int_{k_0^-}^{k_2} f(k) \exp\{-ik[r + 2(s+1)\, x_2(0)]\}\, dk \quad , \tag{10·02}$$

$$I_7 = I_8 = \int_{k_0^-}^{k_2} f(k) \exp\{-ik[r + 2s\, x_2(0)]\}\, dk \quad . \tag{10·03}$$

These integrals have points of stationary phase for $r = 2(s-1)\, r_2(0)$, $2(s+1)\, r_2(0)$, and $2sr_0(0)$, respectively. For $s = 0$, only I_6 has a point of stationary phase; but the same value of r is a point of stationary phase for I_5 with $s = 2$, and for I_7 and I_8 with $s = 1$. Therefore, since

$$e^{-(3\pi i/2)(s+1)}\, e^{-\pi i/2}\, I_{5,s+1} = e^{-(3\pi i/2)(s-1)}\, e^{\pi i/2}\, I_{6,s-1} = e^{-(3\pi i s/2)}\, I_{7,s}$$

Eq. (8·05) becomes

$$\phi = 2A \exp[i(\omega t - \pi/4 - 3s\pi/2)]\, I \quad , \tag{10·04}$$

where I is given by Eq. (10·03) and $s = 1, 2, 3, \ldots$. Further, $R = 2sr_2(0)$ and $T = 2st_2(0)$. Values of $2x_2(0)$, $2r_2(0)$, and $2\partial r_2(0)/\partial\beta$ for various values of β, computed for the velocity-altitude function given above, are given in Table 1.

TABLE 1

Values of Parameters Occurring in I

β (km/sec)	$2x_2(0)$ (km)	$2r_2(0)$ (km)	$2dr_2(0)/d\beta$ (sec)
0.344	36.562	240.63	$-\infty$
.34449	37.154	230.79	−9088
.345	37.344	227.29	−5882
.347	38.852	219.51	−2740
.350	41.070	213.49	−1494
.355	44.668	208.25	−720
.360	48.208	205.65	−358.2
.365	51.710	204.43	−144.6
.370	55.224	204.08	−2.2
.375	58.740	204.33	+99.2
.380	62.248	205.03	+174.4

The infinity in $dr_2(0)/d\beta$ at $\beta = c_0$, which is of an order such that $\tan\theta(0,k)$ $dr_2(0)/d\beta$ is finite, would lead to difficulties for r near $R(k_0)$ if we took the lower limit of integration in Eq. (10·03) to be $k_0 = \omega/c_0$ instead of $k_0^{\cdot}$. Except where this infinity is involved, however, we may take the integration limits to be ω/c_0 and ω/c_2, since these differ very slightly from $k_0^{\cdot}$ and k_2. (For example, for a frequency of 1 cycle, $\beta(k_0^{\cdot}) = 0.34449$ and $\beta(k_2) = 0.37961$ in place of 0.344 and 0.380, respectively.)

From Table 1 we note that $r_2(0)$ has a minimum for $\beta = 0.370$ (nearly). Case III, with $s = 1, 2,$ etc., is therefore the appropriate approximation for values of r near 204, 408, etc. At these minima of R we have $\partial^2 R/\partial\beta^2 \simeq 2.4 \times 10^4 s\ \sec^2 km^{-1}$, from which we find the limits of the integral in Eq. (9·18) are $v_b \simeq -0.553(\nu s)^{1/3}$ and $v_a \simeq 1.589(\nu s)^{1/3}$. Except for quite high frequencies, or large values of s, these quantities are not sufficiently large for the Airy integral with infinite limits to be applicable. The corresponding value of the parameter w is $0.808\ (204.08s - r)\nu^{2/3}\ s^{-1/3}$, and when $|204.08s - r| > 5.69s$, $|w| > |v_a - v_b|^2$. Under these conditions we can obtain a crudely approximate evaluation of the integral in Eq. (9·18) by neglecting the v^3 term in the exponent. The absolute value of this integral is then

$$\left|\int_{v_a}^{v_b} e^{iwv}\,dv\right| = \frac{\{2[1 - \cos w(v_a - v_b)]\}^{1/2}}{|w|}$$

$$= \frac{1.75 s^{1/3}\nu^{-2/3}}{|204.08s - r|}\left\{1 - \cos 1.73\ \nu(204.08s - r)\right\}^{1/2},$$

from which the distance Δr between diffraction maxima is $3.632/\nu$ km and the envelope of the maxima falls off as $2.476 s^{1/3}/|204.08s - r|\nu^{2/3}$.

Case I is the appropriate approximation for the integral I in the ranges $205 + < r < 2r_2(0,k_0^{\cdot})$ with $s = 1$, $410 + < r < 4r_2(0,k_0^{\cdot})$ with $s = 2$, etc. Values of $v_a\,\nu^{-1/2}$ and $v_b\,\nu^{-1/2}$ for this case are given for various values of r in Table 2. From these figures it appears that for frequencies of the order of 1 cycle the limits of the Fresnel integral in Eq. (9·06) are not large, so that diffraction minima and maxima will occur over the whole width of the geometrical wave zone.

Case II is applicable for values of r greater than $2sr_2(0,k_0^{\cdot})$, which for a 1-cycle wave has the values 230.79, 461.58, etc. in the present example. The limits of the Fresnel integral as given by Eqs. (9·16) and (9·17) are both of the same sign, so that the integral

decreases monotonically as r increases. Inserting numerical values for $\nu = 1$, Eqs. (9·16) and (9·17) become

$$v_a = \frac{r - 230.79s}{23.20s^{1/2}} ,$$

$$v_b = \frac{r + 91.6s}{23.20s^{1/2}} .$$

The upper limit, v_b, is thus a fairly large quantity for all values of s, and the value of the Fresnel integral is determined chiefly by v_a.

TABLE 2

Values of Limits of Fresnel Integral in Eq. (9·06) for Various Values of r

r(km)	$v_b/\nu^{1/2}$	$v_a/\nu^{1/2}$
205.65	0.82	−2.06
208.25	1.55	−1.77
213.49	2.57	−1.29
219.51	3.51	−0.81
227.29	4.51	−0.40
240.63	5.91	0.0
411.30	1.16	−2.92
416.50	2.19	−2.50
426.98	3.63	−1.83
439.02	4.96	−1.14
454.58	6.38	−0.57
481.26	8.36	0.0

The resulting values of $|\phi|$ in and near the first geometrical wave zone ($s = 1$) are plotted as a function of r for a frequency of 1 cycle (Fig. 12). The dashed curve shows the "geometrical" value, i.e., the limit approached for infinite frequency. At longer ranges, corresponding to larger values of s, the values of v_a and v_b given by Eqs. (9·07) and (9·08) increase in proportion to $s^{1/2}$, so that the oscillations of the Fresnel integral within the geometrical wave zone become smaller. The limits of integration given by Eqs. (9·20) and (9·21) increase in proportion to $s^{1/3}$, while the parameter w, for a

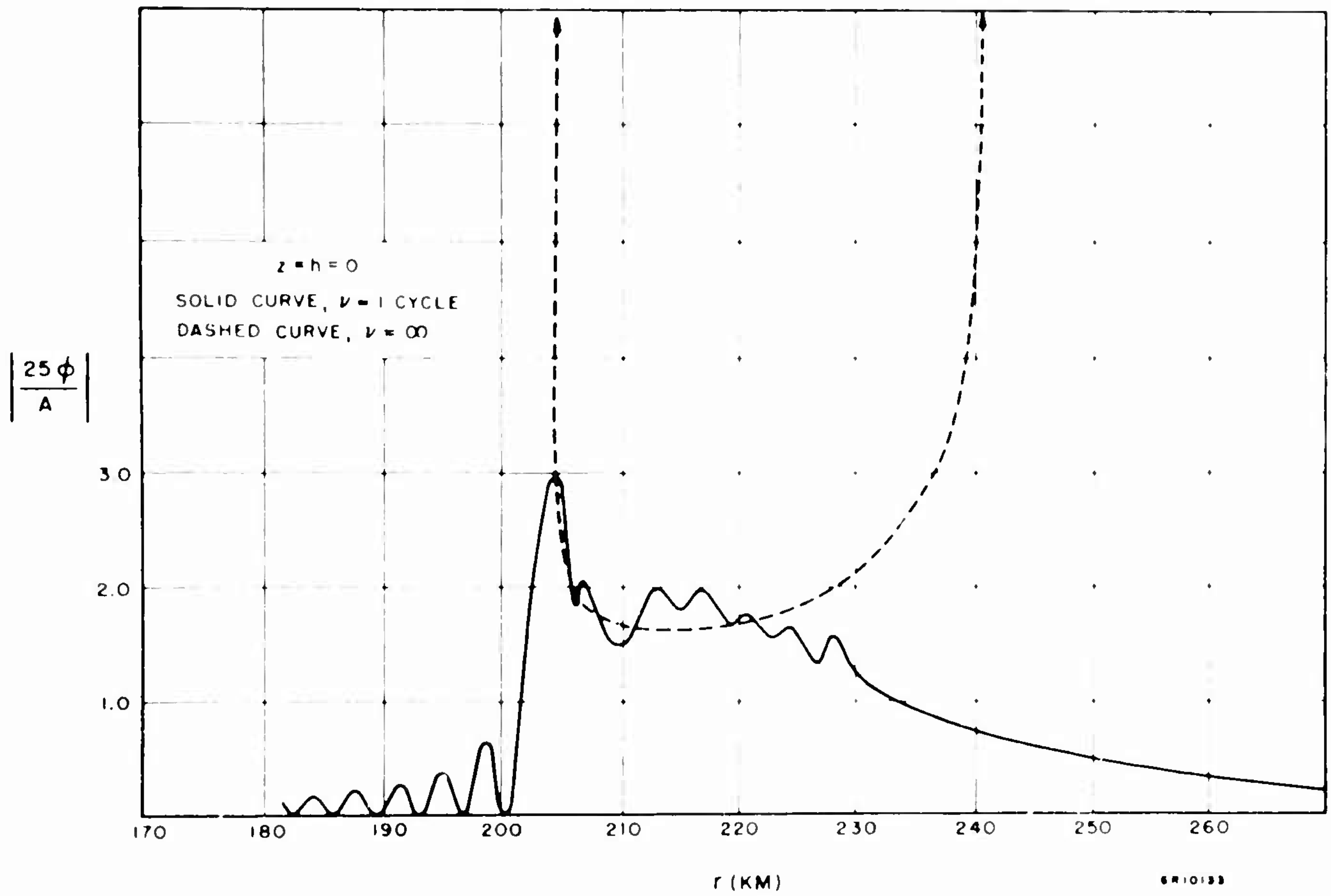

Fig. 12. Relative amplitude distribution in and near first geometric wave zone.

given distance [R (K) − r] from the edge of the geometrical shadow zone, decreases as $s^{-1/3}$. The complete Airy integral thus becomes a better approximation for the integral with finite limits in Eq. (9·18). Ignoring the minor diffraction oscillations, the gross variations of $|\phi|$ with distance at very long ranges are shown in Fig. 13, for a frequency of 1 cycle. The dashed curve shows the value of $|\phi|$ for a source of the same strength in a medium of constant propagation velocity.

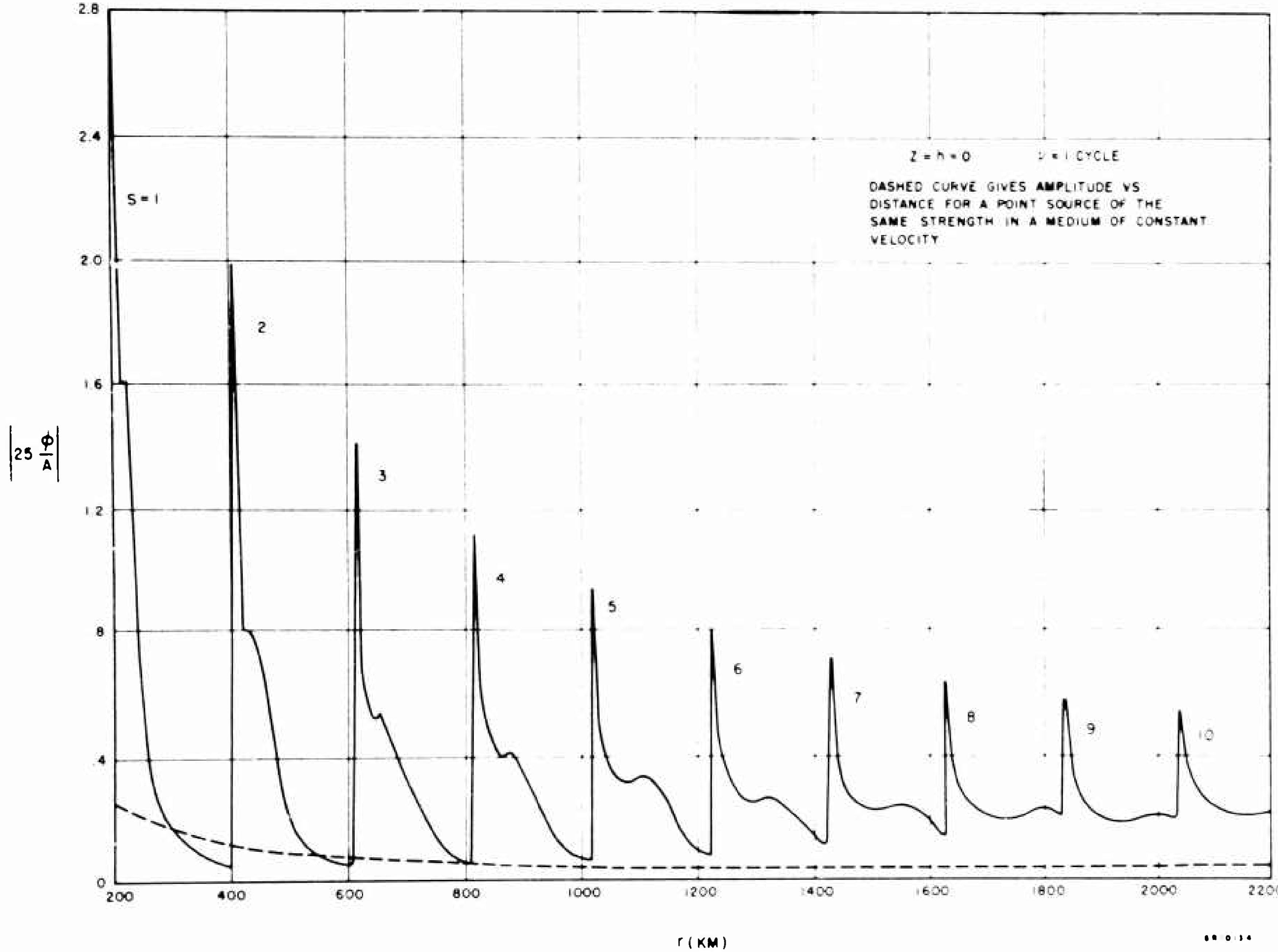

Fig. 13. Relative amplitude at long ranges.